

STABLE AND SINGULAR SOLUTIONS OF THE EQUATION $\Delta u = \frac{1}{u}$

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Abstract: We study properties of the semilinear elliptic equation $\Delta u = \frac{1}{u}$ on domains in \mathbb{R}^n , with an eye toward nonnegative singular solutions as limits of positive smooth solutions. We prove the nonexistence of such solutions in low dimensions when we also require them to be stable for the corresponding variational problem. The problem of finding singular solutions is related to the general study of singularities of minimal hypersurfaces of Euclidean space.

1. INTRODUCTION

One way to obtain singular minimal hypersurfaces with symmetry is given in the paper [14] by Simon. Given positive $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ solving the equation

$$(1) \quad \mathcal{M}u := \sum_{i=1}^n D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = \frac{m}{u \sqrt{1 + |Du|^2}}$$

and an m -dimensional closed subgroup Γ of the orthogonal group in \mathbb{R}^N , with $G_p := \{g(p) : g \in \Gamma\}$ an orbit of maximal volume over $p \in S^{N-1}$, the “symmetric graph” $G(u)$ defined by

$$G(u) = \{(x, u(x)\omega) : x \in \Omega, \omega \in G_p\} \subset \mathbb{R}^{n+N}$$

will be stationary with respect to $n+m$ -dimensional volume, and will have the same regularity as the function u . Positive solutions $u > 0$ are smooth. However, if a sequence u_j of such solutions converges continuously to a weak solution $u \geq 0$, then u will be singular exactly at the points where it is zero. The corresponding $G(u)$ will be a singular minimal submanifold. A degree theoretic program for obtaining such sequences of solutions is outlined in Simon’s paper. For more on equation (1), see the survey paper by Dierkes [6]. Our goal is to apply a similar program to the equation $\Delta u = \frac{1}{u}$. The basic degree argument is presented in Section 4.

Notice that if we linearize the left hand side of (1) in the form

$$\sum_i D_i D_i u - \sum_{i,j} \frac{D_i u D_j u D_i D_j u}{1 + |Du|^2} = \frac{m}{u},$$

the resulting equation is $\Delta u = \frac{m}{u}$. For this equation the constant m scales with the independent variable, i.e. $u(Cx)$ solves $\Delta u = \frac{C^2 m}{u}$, so we may

restrict our attention to the case $m = 1$:

$$(2) \quad \Delta u = \frac{1}{u}$$

Both equations (2) and (1), with $m = 1$, have the particular solution $u(x) = C|x|$ with $C = 1/\sqrt{n-1}$. By the results of section 3, this solution is indeed the limit of a sequence of positive smooth solutions.

We note that equation (2) is the Euler–Lagrange equation for the variational integral

$$(3) \quad \mathcal{F}(u) = \int_{\Omega} \frac{1}{2} |Du|^2 + \log u$$

while equation (1) has variational integral

$$\int_{\Omega} u^m \sqrt{1 + |Du|^2},$$

which is the $n + m$ -dimensional volume of $G(u)$. Notice that for arbitrary positive u , the integral (3) is not bounded below for any positive boundary data φ on $\partial\Omega$. Indeed the function $u_\epsilon := \varphi + \zeta(\epsilon - \varphi)$, where ζ is a smooth cutoff function on Ω , satisfies $\mathcal{F}(u_\epsilon) \rightarrow -\infty$ as $\epsilon \rightarrow 0$.

Equation (2) also arises in relation to chemical catalyst kinetics (See [1] and [5]). Here it is a special case of the more general equation

$$(4) \quad \Delta u = u^{-\alpha}, \quad \alpha > 0$$

For $\alpha \neq 1$, (4) is stationary for the variational integral

$$(5) \quad \int_{\Omega} \frac{1}{2} |Du|^2 + \frac{1}{1-\alpha} u^{1-\alpha}$$

Note that when $0 < \alpha < 1$, this integral is bounded below, and by lower semicontinuity, as in [11], minimizers among nonnegative functions with given boundary data exist. In fact, most results concerning (4), as in [5], are limited to the case $0 < \alpha < 1$ and are results on minimizers of the variational problem. In this way, (2) is an interesting limiting case.

An important difference in our method is that the singular limit must (weakly) satisfy the PDE $\Delta u = \frac{1}{u}$ on the whole domain Ω . Minimizers of (5) need not satisfy the Euler–Lagrange equation (4) on the whole domain, in fact they need only solve the *free boundary problem* on the set $\{u > 0\}$.

Despite the nonexistence of minimizers, we may consider solutions u of (2) which are also stable for $\mathcal{F}(u)$ in the sense that $\left. \frac{d^2}{dt^2} \mathcal{F}(u + t\zeta) \right|_{t=0} \geq 0$ for test functions ζ . In this case, u satisfies the stability inequality

$$(6) \quad \int_{\Omega} \frac{\zeta^2}{u^2} \leq \int_{\Omega} |D\zeta|^2$$

for all test functions ζ . We call such u “stable solutions.”

In Sections 7, 8, and 9, we prove the main results on stable solutions, namely, the Hölder continuity of stable solutions, the nonexistence of singular stable solutions in dimension less than seven, and an estimate on the

size of the singular set of a stable solution. The existence and uniqueness of stable solutions is presented in Sections 5 and 6. Here we state the main results.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ and suppose there is a subsolution v with $\Delta v \geq 1/v$ on Ω with boundary values φ_0 on $\partial\Omega$. Then for any $\varphi \geq \varphi_0$, there is a unique stable solution u of $\Delta u = 1/u$ on Ω with boundary values φ .*

Theorem 2. *For a stable solution u with boundary data $\varphi \leq M$, and for every $0 < \alpha < 1$ and $\tilde{\Omega} \subset\subset \Omega$, there is a constant $C(n, M, \alpha, \tilde{\Omega})$ such that for all $x, y \in \tilde{\Omega}$, $|u(x) - u(y)| \leq C|x - y|^\alpha$.*

Theorem 3. *Let $2 \leq n \leq 6$ and let u be a positive stable solution of $\Delta u = \frac{1}{u}$ on the $C^{1,1}$ domain Ω , with boundary data $\varphi \in C^{2,\alpha}(\Omega)$, $|\varphi|_{2,\alpha} \leq M$, and $\varphi \geq \epsilon > 0$. Then there is a constant $\delta = \delta(\Omega, M, \epsilon)$ such that $u \geq \delta$ on Ω .*

Corollary 4. *In dimension less than seven, there are no singular stable solutions of (2).*

Theorem 5. *Suppose u is a limit of positive stable solutions of $\Delta u = \frac{1}{u}$ on a domain Ω with singular set $A = \{u = 0\}$. Then the Hausdorff dimension $\dim_{\mathcal{H}}(A) \leq n - 4 - 2\sqrt{2}$.*

2. BASIC FACTS

Positive solutions to equation (2) are subharmonic, so the maximum principle implies that they achieve their maximum on the boundary of Ω . However, the difference $w = u - v$ of two solutions satisfies the equation $\Delta w + \frac{w}{uv} = 0$, and so the maximum principle does not guarantee that a positive solution is a unique solution of the Dirichlet problem for its boundary data. Indeed, solutions of the analogous ordinary differential equation $u'' = \frac{1}{u}$ need not be unique, and positive radially symmetric solutions in low dimension may not be unique.

Solutions are invariant under homothetic scaling of the graph. That is, if $u(x)$ is a solution on Ω , then $u(Cx)/C$ is also a solution on the domain Ω/C for $C > 0$. In particular, the conical solution scales under homothety of the graph to itself, when centered at the origin.

Nonnegative limits of positive solutions are singular, as in the following

Lemma 6. *If $u \geq 0$ is a weak solution of $\Delta u = \frac{1}{u}$ in a neighborhood of x_0 with $u(x_0) = 0$, then u is not differentiable at x_0 .*

Proof. Suppose we have a weak solution u with $u \geq 0$, $u(x_0) = 0$, and u differentiable at x_0 . Then $Du(x_0) = 0$, and for any $\epsilon > 0$, in a small enough ball $B = B_\delta(x_0)$ we have $0 \leq u \leq \epsilon|x - x_0|$. The weak equation in this ball is

$$(7) \quad \int_B u \Delta \zeta = \int_B \frac{\zeta}{u}$$

where ζ is C^2 with compact support in the ball.

We choose ζ radially symmetric such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B_{\delta/2}$, and $|\Delta\zeta| \leq \frac{C}{\delta^2}$. Then

$$\int_B \frac{\zeta}{u} \geq \int_{B_{\delta/2}} \frac{1}{\epsilon|x|} = \frac{n\omega_n}{(n-1)\epsilon} \left(\frac{\delta}{2}\right)^{n-1}$$

where ω_n is the volume of the unit n -ball. But also

$$\int \frac{\zeta}{u} = \int_{B_\delta} u \Delta \zeta \leq \int_{B_\delta} \epsilon|x| \frac{C_1}{\delta^2} = \frac{n\omega_n C_2 \epsilon}{n+1} \delta^{n-1}.$$

Thus, $\epsilon^2 \geq \left(\frac{n+1}{n-1}\right) \frac{2^{1-n}}{C}$, a contradiction for ϵ small enough.

Using similar methods, we can derive some basic positivity results.

Lemma 7. *Suppose u is a positive and smooth subsolution, i.e. $\Delta u \geq \frac{1}{u}$, on a ball $B_{2\rho}$. Then the following hold:*

- (1) $\frac{1}{\rho^2} \int_{B_{2\rho} \setminus B_\rho} u^2 \geq \omega_n \rho^n$
- (2) $\sup_{B_{2\rho}} u \geq \frac{\rho}{\sqrt{2^n - 1}}$

Notice that the second property says that there do not exist solutions on any ball that are uniformly small, and any solution defined on all of \mathbb{R}^n must be unbounded.

3. ASYMPTOTICALLY CONICAL AND RADIAL SOLUTIONS

In this section it will be convenient to consider the rescaled equation

$$\Delta u = \frac{n-1}{u}$$

which has the conical solution $u(x) = |x|$. We will also use the radial variable $r = |x|$.

Caffarelli, Hardt, and Simon proved in [2] the existence of minimal surfaces asymptotic to minimal cones. Following their proof we get a similar result showing the existence of a wide variety of singular solutions of (2) on the ball which are asymptotic to our conical solution at the origin.

The wide variety of singular solutions comes from the existence of solutions with boundary data which are small perturbations of constant boundary data equal to one on the sphere ∂B_1 . As in [2], we are only able to specify the perturbed boundary data of the asymptotic solution in the orthogonal complement of a finite dimensional subspace of $L^2(\partial B_1)$, which depends on the rate at which our solutions will be asymptotic to the cone. In this case, the finite dimensional subspace is the span of the first J eigenvectors of the operator $-\Delta - (n-1)$ on the sphere, where the remaining eigenvectors have eigenvalues μ large enough that $1 - \frac{n}{2} + \sqrt{\frac{(n-2)^2}{4} + \mu} > m$. The projection onto the orthogonal complement is denoted Π_J . The result may be expressed in terms of the scaled Hölder norm on annuli, defined by

$$|f|_{2,\alpha;r} = \sum_{l=0}^2 r^l \sup_{r \leq |x| \leq 2r} |D^l u| + r^{k+\alpha} \sup_{\substack{x \neq y \\ r \leq |x|, |y| \leq 2r}} \frac{|D^2 f(x) - D^2 f(y)|}{|x-y|^\alpha}.$$

Theorem 8. Given $m > 1$ and $0 < \alpha < 1$, there exist ϵ and C depending on m, n , and α so that for any function g on ∂B_1 with $|g|_{C^{2,\alpha}} < \epsilon$, there exists a solution u of $\Delta u = \frac{n-1}{u}$ on B_1 with $\Pi_J(u-1) = \Pi_J g$ on ∂B_1 and satisfying for $0 < r < 1/2$

$$r^{-m} |u - |x||_{2,\alpha;r} \leq C |g|_{C^{2,\alpha}}.$$

The proof of this theorem is essentially identical to the argument given in [2].

We call smooth positive solutions of $\Delta u = \frac{n-1}{u}$ which are radially symmetric, i.e. $u(x) = u(r)$, $r = |x|$, “radial solutions.” The resulting ordinary differential equation satisfied by u is $u_{rr} + \frac{n-1}{r} u_r - \frac{n-1}{u} = 0$, with the particular solution $u = r$. The following two theorems on radial solutions are the most useful for further analysis of the PDE. Theorem 9 is due to Brauner–Nicolaenko [1], using bifurcation theory in the context of equation (4). We have alternative proofs of these and more general facts using basic ODE techniques.

Theorem 9. For any $\epsilon > 0$, solutions of the ODE problem $u_{rr} + \frac{n-1}{r} u_r - \frac{n-1}{u} = 0$ with $u(0) = \epsilon$ and $u'(0) = 0$ exist uniquely on $[0, \infty)$. These solutions satisfy $u(r) - r = O(1)$, and consequently as $\epsilon \rightarrow 0$, the solutions $u(r) \rightarrow r$ uniformly on compact subsets.

Theorem 10. There exist constants C_1 and C_2 depending on n such that on the ball $B_1(\mathbf{0})$, the Dirichlet problem

$$\begin{aligned} \Delta u &= \frac{n-1}{u} && \text{on } B_1 \\ u &= C && \text{on } \partial B_1 \end{aligned}$$

has a solution for $C > C_1$, and has a unique solution for $C > C_2$. For $n \geq 7$, $C_1 = C_2 = 1$.

Thus, the radial conic solution $u = |x|$ is indeed a limit of positive smooth solutions. An interesting further result is that these conic solutions are stable for $n \geq 7$ and unstable for $2 \leq n \leq 6$. We state the result for the original equation $\Delta u = 1/u$.

Lemma 11. The conical solutions $u = \frac{|x|}{\sqrt{n-1}}$ are stable for $n \geq 7$ and are unstable for $2 \leq n \leq 6$.

Proof: This follows from the Hardy inequality with best constant

$$\frac{(n-2)^2}{4} \int_{\Omega} \frac{\zeta^2}{|x|^2} \leq \int_{\Omega} |D\zeta|^2$$

for all $\zeta \in C_c^1(\Omega)$. See [10], or for a simple proof see [7].

4. DEGREE CONSTRUCTION

We will use the Leray–Schauder degree with several different setups. For the basic theory of the degree on Banach spaces, see [4]. In general we will use the Banach space $\mathcal{B} = C^{2,\alpha}(\bar{\Omega})$ and open set $\mathcal{U} = \{u \in \mathcal{B} : u > g, |u|_{2,\alpha} < M_\delta\}$ where g is a fixed positive bounded function with positive minimum δ . A typical operator $T : [0, 1] \times \mathcal{U} \rightarrow \mathcal{B}$ will be defined by $T_t(u) = v$, where v is the solution of

$$\begin{cases} \Delta v = \frac{v}{u^2} & \text{on } \Omega \\ v = \varphi_t & \text{on } \partial\Omega \end{cases}$$

and φ_t are boundary data continuous in t with $g < \varphi_t < M_\delta$. We use the notation $\deg(I - T_t, \mathcal{U}, 0)$ for the Leray–Schauder degree invariant for fixed points of T_t .

In the following results, all solutions are assumed to be positive. Lemma 12 comes from a basic Schauder estimate.

Lemma 12. *For each $0 < \delta < 1$, M_δ can be chosen such that any solution u of $\Delta u = \frac{1}{u}$ with $\delta \leq u \leq \frac{1}{\delta}$ satisfies $|u|_{2,\alpha} < M_\delta$.*

Lemma 13. *If $\varphi_0 \geq C_2(n)$ is constant boundary data on the unit ball in \mathbb{R}^n , $C_2(n)$ as in Theorem 10, and if \mathcal{U} is convex containing φ_0 and the solution of (2) with data φ_0 , then $\deg(I - T_0, \mathcal{U}, 0) = 1$.*

Proof of Lemma 13:

By Theorem 10, the radial solution with $u = \varphi_0$ on ∂B_1 is unique. We let $T_t(u) = v$ be the solution to the problem

$$\begin{cases} \Delta v = \frac{(1-t)v}{u^2} & \text{on } B_1 \\ v = \varphi_0 & \text{on } \partial B_1 \end{cases}$$

so that the unique solution u above is the unique fixed point of T_0 . From u we can scale to $\tilde{u} = u(\sqrt{1-t}x)$ which satisfies $\Delta \tilde{u} = \frac{1-t}{\tilde{u}}$ and is unique relative to its boundary data, $\tilde{u} = u(\sqrt{1-t}) < \varphi_0$ on ∂B_1 . We may then geometrically scale \tilde{u} to get \hat{u} uniquely solving $\Delta \hat{u} = \frac{1-t}{\hat{u}}$ and $\hat{u} = \varphi_0$ on ∂B_1 . Note that $\hat{u} > u$ on B_1 . Thus, T_t has a unique fixed point for all t , and in our Leray–Schauder degree setup, there are no fixed points of T_t on the boundary of \mathcal{U} for any convex \mathcal{U} containing u and φ_0 . So, $\deg(I - T_0, \mathcal{U}, 0) = \deg(I - T_1, \mathcal{U}, 0)$. But $T_1 \equiv C$. So, $\deg(I - T_1, \mathcal{U}, 0) = \deg(I - C, \mathcal{U}, 0) = \deg(I, \mathcal{U}, C) = 1$.

Lemma 14. *There exists $\epsilon(\Omega)$ such that if $\varphi_1 \leq \epsilon$ is boundary data on a domain Ω in \mathbb{R}^n , then no solution with boundary data φ_1 exists and $\deg(I - T_1, \mathcal{U}, 0) = 0$.*

Proof: Lemma 14 follows easily from part 2 of Lemma 7

Lemma 15. *If Ω is an arbitrary domain and the function g is chosen to be the maximum of a finite collection of subsolutions, i.e. $\mathcal{U} = \{u \in C^{2,\alpha} : u > g_1, \dots, u > g_k, |u|_{2,\alpha} < M_\delta\}$ with $\Delta g_k \geq \frac{1}{g_k}$, and if $\varphi_1 > \max g_k$ is boundary data on Ω , then $\deg(I - T_1, \mathcal{U}, 0) = 1$.*

Proof: Consider again the map $T_t(u) = v$ where v is the solution of

$$\begin{cases} \Delta v = \frac{tv}{u^2} & \text{on } \Omega \\ v = \varphi_0 & \text{on } \partial\Omega \end{cases}$$

Since T_0 is constant, it has degree one. Suppose for some t that T_t has a fixed point u in $\partial\mathcal{U}$. Then $\Delta u = \frac{t}{u}$, and for some g_j , we have $u \geq g_j$ and $u(x_0) = g_j(x_0)$ for some x_0 . Then,

$$\Delta(u - g_j) \leq \frac{tg_j - u}{ug_j} \leq 0 \quad u - g_j \geq 0 \text{ on } \Omega.$$

So $u - g_j$ has a zero minimum, contradicting the Hopf Maximum Principle. Thus, $\deg(I - T_1, \mathcal{U}, 0) = 1$.

We now outline a general method for producing “singular sequences” of positive solutions to (2) with minimum tending to zero. In the application of the degree, let us choose $\Omega = B_1 \subset \mathbb{R}^n$. Let $g = \delta_j > 0$ so that $\mathcal{U} = \{u \in C^{2,\alpha} : u > \delta_j, |u|_{2,\alpha} < M_{\delta_j}\}$, and let $\delta_j \searrow 0$. Using Lemmas 13 and 14, we may take φ_t to be any homotopy of boundary data between $\varphi_0 = C \geq C(n)$ a large constant and $\varphi_0 \leq \epsilon$ small. Then since $\deg(I - T_0, \mathcal{U}, 0) = 1$ and $\deg(I - T_1, \mathcal{U}, 0) = 0$, there must exist $t_j \in (0, 1)$ and a fixed point $u_j \in \partial\mathcal{U}$ which solves

$$\begin{cases} \Delta u_j = \frac{1}{u_j} & \text{on } B_1 \\ u_j = \varphi_{t_j} & \text{on } \partial B_1 \end{cases}$$

with $\min_{B_1} u_j = \delta_j$. Then the sequence u_j is a “singular sequence” in the sense that $\min_{B_1} u_j \rightarrow 0$. If $u_j \rightarrow u$ uniformly with $u \geq 0$ and $\min u = 0$, then u is a singular solution. Notice that if $\varphi_t \in C^1$ is bounded, then we at least have a subsequence j' and a t_0 with $\varphi_{t_{j'}} \rightarrow \varphi_{t_0}$. However, we do not yet have the necessary continuity estimates on u_j to get a singular limit u .

In the case $n = 2$, since the u_j are in particular subharmonic, we can use the “log trick” (See Corollary 24 below) to show that

$$\int_{B_\rho} |Du_j|^2 \leq \frac{C}{|\log \rho|}$$

uniformly as $\rho \rightarrow 0$, which is just short of a modulus of continuity estimate and also shows the u_j are uniformly of vanishing mean oscillation as in [19]. Even in dimension 2, subharmonicity cannot be sufficient for a continuity estimate. For example, the functions $u_\epsilon(x) = |x|^\epsilon$ for $\epsilon > 0$ are nonnegative, uniformly bounded on $B_1 \subset \mathbb{R}^2$, and subharmonic, yet do not satisfy any continuity estimate.

5. GENERAL FACTS ABOUT MAXIMAL SOLUTIONS

A maximal solution u of $\Delta u = \frac{1}{u}$ satisfies the property that $v \leq u$ for any other solution v with the same boundary data as u . We show in Lemma 17 below that whenever a subsolution exists for fixed boundary data, there is also a maximal solution with that boundary data. It turns out that the

maximal solution is also the unique stable solution. The existence of maximal solutions can be achieved by the usual method of sub/supersolutions (see [5]). We give an alternative degree method.

We will consistently use the Leray–Schauder degree with the operator $T_t(u) = v$, where v is the solution of

$$\begin{cases} \Delta v = \frac{v}{u^2} \\ v|_{\partial\Omega} = \varphi_t \end{cases}.$$

Lemma 16. *For a nonempty finite set of positive subsolutions u_j with boundary data φ_1 , there is a solution u with boundary data φ_1 such that $u \geq u_j$ for all j .*

Proof. Consider the open set

$$\mathcal{U} = \bigcap_j \{u \in C^{2,\alpha} : u(x) > u_j(x), |u|_{2,\alpha} < M_\delta\}$$

in $C^{2,\alpha}(\Omega)$, where all $u_j > \delta$ and M_δ is chosen according to Lemma 12. Take data $\varphi_0 > \varphi_1$ and let φ_t be any smooth decreasing homotopy from φ_0 to φ_1 . By Lemma 15, $\deg(I - T_t, \mathcal{U}, 0) = 1$ for all $t < 1$, and thus there exist solutions $u_t \in \mathcal{U}$ for all $t < 1$. Consider any sequence $t_j \nearrow 1$, and corresponding solutions u_{t_j} . Since all of these functions are uniformly bounded below, the Schauder estimates give us a uniform C^3 bound, so by Arzela–Ascoli, a subsequence $u_{t_k} \rightarrow u \in \bar{\mathcal{U}}$ in $C_{2,\alpha}$, and $\Delta u = 1/u$, $u|_{\partial\Omega} = \varphi_1$, $u \geq u_j \forall j$.

Lemma 17. *If there is a positive subsolution u_0 to $\Delta u = \frac{1}{u}$ on Ω with boundary data φ_0 , and if $\varphi \geq \varphi_0$, then there is a unique maximal solution with boundary data φ .*

Proof. Consider the collection \mathcal{C} of all solutions $u \geq u_0$ with boundary data φ . By Lemma 15, this collection is nonempty. \mathcal{C} is partially ordered by the relation $u_\alpha \leq u_\beta$ on Ω . By the Hausdorff Maximality Theorem, there exists a maximal totally ordered subset S . For any $x_0 \in \Omega$ let $u_\alpha, u_\beta \in S$ with $u_\alpha \neq u_\beta$ and $u_\alpha \leq u_\beta$. By the maximum principle, we have $u_\alpha(x_0) < u_\beta(x_0)$. Thus, S can be indexed by $u_\alpha(x_0)$. That is, $S = \{u_\alpha\}_{\alpha \in A}$ where $\alpha = u_\alpha(x_0)$. By the maximum principle, since $\Delta u_\alpha \geq 0$, $u_\alpha \leq \sup \varphi$ for all α , so A is bounded above. Let $\alpha_\infty = \sup_A \alpha$. We claim that $\alpha_\infty \in A$, and u_{α_∞} is a maximal solution. Consider $u_{\alpha_j} \in S$ with $\alpha_j \nearrow \alpha_\infty$ and $u_{\alpha_j} \leq u_{\alpha_{j+1}}$. Since these are uniformly bounded below and monotone increasing, the Schauder estimates and Arzela–Ascoli give a function u_{α_∞} with $u_{\alpha_j} \nearrow u_{\alpha_\infty}$, where $u_{\alpha_\infty}(x_0) = \alpha_\infty$ and u_{α_∞} is a solution. For any $u_\alpha \in S$, choose j large so that $\alpha_j > \alpha$. Then $u_{\alpha_j} \geq u_\alpha$ by total ordering, and $u_{\alpha_\infty} \geq u_{\alpha_j}$ since the sequence was monotone. Thus, u_{α_∞} is an upper bound for S , so by maximality $u_{\alpha_\infty} \in S$. To see that u_{α_∞} is a maximal solution, suppose v is another solution with $v(x_1) > u_{\alpha_\infty}(x_1)$ for some $x_1 \in \Omega$. By Lemma 16,

there exists a solution \tilde{v} with $\tilde{v} \geq u_{\alpha_\infty}$ and $\tilde{v} \geq v$. Also by the hypothesis on x_1 , $\tilde{v} \neq u_{\alpha_\infty}$. But then $S \cup \{\tilde{v}\}$ is totally ordered, contradicting maximality.

Lemma 18. *If $\varphi_0 < \varphi_1$, and u_0, u_1 are maximal solutions with boundary data φ_0 and φ_1 respectively, then $u_0 < u_1$.*

Proof. By Lemma 15, there exists a solution u to $\Delta u = \frac{1}{u}$ with data φ_1 and $u > u_0$, since u_0 is a subsolution. Since u_1 is maximal, $u_1 \geq u > u_0$.

Lemma 19. *If u is a maximal solution on Ω and $\tilde{\Omega} \subset \Omega$ is a subdomain with continuous boundary, then u restricted to $\tilde{\Omega}$ is a maximal solution with respect to its boundary data on $\partial\tilde{\Omega}$.*

Proof. Suppose not. Then there is a maximal solution v on $\tilde{\Omega}$ with boundary data u and $v > u$ on $\tilde{\Omega}$. Let \mathcal{U} be the open set

$$\mathcal{U} = \left\{ w \in C^{2,\alpha} : w > u \text{ on } \Omega, w > v \text{ on } \tilde{\Omega}, |w|_{2,\alpha} < M_\delta \right\}.$$

Let φ_1 be any boundary data on $\partial\Omega$ greater than the boundary data φ_0 of u . As in the proof of Lemma 15, consider the operator $T_t(w) = \tilde{w}$ where \tilde{w} is the solution of

$$\begin{cases} \Delta \tilde{w} = \frac{t\tilde{w}}{\tilde{w}^2} & \text{on } \Omega \\ \tilde{w} = \varphi_1 & \text{on } \partial\Omega \end{cases}$$

Suppose w is a fixed point of T_t in $\bar{\mathcal{U}}$. By the Hopf Maximum Principle, $w > u$ on Ω . Thus, $w > u$ on $\partial\tilde{\Omega}$. Then we may apply the maximum principle on $\tilde{\Omega}$, so $w > v$ on $\tilde{\Omega}$. Thus, w cannot be on the boundary of \mathcal{U} . So, $\deg(I - T_1, \mathcal{U}, 0) = \deg(I - T_0, \mathcal{U}, 0) = 1$. Then, for any boundary data $\varphi_t > \varphi_0$, there exists a solution w of $\Delta w = \frac{1}{w}$ on Ω with $w = \varphi_t$ on $\partial\Omega$, $w > u$ on Ω , and $w > v$ on $\tilde{\Omega}$. Now we let φ_t be any smooth decreasing homotopy of boundary data approaching φ_0 . Let w_t be the corresponding solutions whose existence we just proved. By the Schauder estimates and Arzela–Ascoli, there exists a sequence w_{t_j} with $t_j \rightarrow 0$ such that w_{t_j} converges to a solution w with boundary data φ_0 , and with $w \geq u$ on Ω and $w \geq v$ on $\tilde{\Omega}$. Thus, $w \geq v > u$ on $\tilde{\Omega}$, contradicting the maximality of u .

6. STABILITY OF MAXIMAL SOLUTIONS

Recall that stable solutions satisfy the stability inequality (6).

Lemma 20. *Maximal solutions of $\Delta u = 1/u$ are stable.*

Proof. Let u_0 be a maximal solution with data φ_0 , and let $\varphi_t = \varphi_0 + t$ for $t > 0$. By Lemma 17, there exist maximal solutions u_t with data φ_t and $u_t > u_0$. By the Schauder estimates the u_t are also bounded in C^4 . For a sequence $t_j \searrow 0$, we then have a subsequence such that $u_{t_j} \rightarrow \tilde{u}_0$ in $C^{2,\alpha}$,

with $\tilde{u}_0 \geq u_0$, and by maximality $\tilde{u}_0 = u_0$. Let $\delta_j = \max_{\Omega}(u_{t_j} - u_0)$, and let $v_j = \frac{u_{t_j} - u_0}{\delta_j}$. Then

$$\Delta v_j = \frac{u_0 - u_{t_j}}{u_0 u_{t_j} \delta_j}.$$

By the Schauder estimates, v_j is bounded in $C^{2,\alpha}$, so by Arzela–Ascoli, a subsequence $v_j \rightarrow v$ in C^2 . The function v is nonnegative, not identically 0, has nonnegative boundary data, and satisfies the linearized equation $\Delta v + \frac{1}{u_0^2}v = 0$. By the maximum principle, $v > 0$ in Ω . The weak equation for v is then

$$\int_{\Omega} Dv \cdot D\zeta = \int_{\Omega} \frac{\zeta v}{u_0^2}.$$

We use the test function $\frac{\zeta^2}{v}$ for ζ , and Cauchy–Schwartz to get the desired inequality

$$\int \frac{\zeta^2}{u_0^2} \leq \int |D\zeta|^2$$

for all compactly supported ζ .

In fact, the maximal solution for given boundary data is the only stable solution.

Lemma 21. *The maximal solution for given boundary data φ is the unique stable solution with data φ .*

Proof: Let u be the maximal solution and let v be any other positive solution. Then $u = v + w$ where $w > 0$ in Ω and $w = 0$ on $\partial\Omega$. Thus,

$$\Delta w = \frac{1}{v+w} - \frac{1}{v} = \frac{-w}{v(v+w)} > \frac{-w}{v^2}$$

and so, integrating by parts with w ,

$$\int_{\Omega} \frac{w^2}{v^2} > \int_{\Omega} w \Delta w = \int_{\Omega} |Dw|^2$$

and v cannot be stable.

Theorem 1 now follows from Lemmas 17, 20, and 21

7. HÖLDER CONTINUITY OF STABLE SOLUTIONS

We now prove a main result that stable solutions are locally uniformly Hölder continuous.

Theorem 22. *For a stable solution u with boundary data $\varphi \leq M$, and for every $0 < \alpha < 1$ and $\tilde{\Omega} \subset \subset \Omega$, there is a constant $C(n, M, \alpha, \tilde{\Omega})$ such that for all $x, y \in \tilde{\Omega}$, $|u(x) - u(y)| \leq C|x - y|^\alpha$.*

Proof. Let u be a smooth positive stable solution on Ω . The weak form of the equation is

$$(8) \quad \int Du \cdot D\zeta = - \int \frac{\zeta}{u}$$

for ζ compactly supported. Here and throughout the rest of the proof, all integrals are taken over the domain Ω . Substituting $u\zeta^2$ for ζ in (8) yields

$$\int |Du|^2 \zeta^2 + 2 \int u\zeta Du \cdot D\zeta = - \int \zeta^2$$

and applying the Cauchy–Schwartz inequality, we get

$$(9) \quad \int \zeta^2 \left(\frac{1}{2} |Du|^2 + 1 \right) \leq 2 \int u^2 |D\zeta|^2.$$

Substituting $\frac{\zeta^2}{u}$ in (8) and again using Cauchy–Schwartz gives

$$(10) \quad \int \frac{|Du|^2}{u^2} \zeta^2 \leq 2 \int \frac{\zeta^2}{u^2} + 4 \int |D\zeta|^2.$$

Differentiating the equation with respect to x_l , and using subscripts to denote differentiation, we have $\Delta u_l = -\frac{u_{ll}}{u^2}$, and thus the weak equation

$$\int u_{li} \zeta_i = \int \frac{u_l}{u^2} \zeta,$$

where we sum on the repeated index i . Note that this equation is equivalent to using ζ_l in (8) and integrating by parts. Substituting $u_l \zeta^2$ for ζ gives

$$\int u_{li} u_{li} \zeta^2 + 2 \int u_{li} u_l \zeta \zeta_i = \int \frac{u_l^2}{u^2} \zeta^2,$$

or, after summing on l ,

$$(11) \quad \int |D^2 u|^2 \zeta^2 + 2 \int u_l u_{li} \zeta \zeta_i = \int \frac{|Du|^2}{u^2} \zeta^2$$

It is now convenient to use the variable $v = \sqrt{1 + |Du|^2}$, where

$$v_i = \frac{1}{v} u_j u_{ji}, \quad |v_i|^2 \leq \sum_j |u_{ij}|^2,$$

and

$$|Dv|^2 \leq \sum_{ij} |u_{ij}|^2 = |D^2 u|^2.$$

Also, $vv_i = u_l u_{li}$. Replacing in equation (11) gives

$$(12) \quad \int |Dv|^2 \zeta^2 + 2 \int vv_i \zeta \zeta_i \leq \int \frac{|Du|^2}{u^2} \zeta^2.$$

If u is stable, it additionally satisfies (6),

$$\int \frac{\zeta^2}{u^2} \leq \int |D\zeta|^2.$$

Now we can combine our inequalities:

$$\begin{aligned} \int |Dv|^2 \zeta^2 + 2 \int v v_i \zeta \zeta_i &\leq \int \frac{|Du|^2}{u^2} \zeta^2 && \text{by (12)} \\ &\leq 2 \int \frac{\zeta^2}{u^2} + 4 \int |D\zeta|^2 && \text{by (10)} \\ &\leq 6 \int |D\zeta|^2 && \text{by (6)} \end{aligned}$$

and we get the main inequality for stable solutions:

$$(13) \quad \int |Dv|^2 \zeta^2 + 2 \int v v_i \zeta \zeta_i \leq 6 \int |D\zeta|^2$$

We use this and the Sobolev Inequality to iteratively estimate integrals $\int v^q \zeta^\beta$ for $q \geq 0$. In fact, the estimate for $q = 2$ and $\beta = 2$ is contained in (9). For $q > 2$, replace ζ in (13) by $v^q \zeta$ to get

$$\begin{aligned} \int |Dv|^2 v^{2q} \zeta^2 + 2q \int |Dv|^2 v^{2q} \zeta^2 + 2 \int v^{2q+1} \zeta v_i \zeta_i &\leq \\ &\leq 6 \int |qv^{q-1} \zeta Dv + v^q D\zeta|^2 \end{aligned}$$

and so

$$(14) \quad \begin{aligned} \int |Dv|^2 v^{2q} \zeta^2 + 2q \int |Dv|^2 v^{2q} \zeta^2 &\leq \\ 2 \int |Dv| |D\zeta| v^{2q+1} \zeta + 12q^2 \int |Dv|^2 v^{2q-2} \zeta^2 + 12 \int v^{2q} |D\zeta|^2 & \end{aligned}$$

where we have used the squared triangle inequality $(a+b)^2 \leq 2a^2 + 2b^2$. We use the Cauchy–Schwartz inequality to eliminate the second term on the top line of (14) with the first term on the bottom line.

$$(15) \quad \begin{aligned} \int |Dv|^2 v^{2q} \zeta^2 &\leq \frac{1}{2q} \int v^{2q+2} |D\zeta|^2 + 12 \int v^{2q} |D\zeta|^2 + \\ &\quad + 12q^2 \int |Dv|^2 v^{2q-2} \zeta^2 \end{aligned}$$

Notice at this point that the last term on the right side of the inequality is the same as the left hand side with a lower power of v . So, we can apply (15) to that term iteratively until $2q - 2$ is less than zero, and use the fact that $v \geq 1$ to get

$$(16) \quad \int |Dv|^2 v^{2q} \zeta^2 \leq C(n, q) \int v^{2q+2} |D\zeta|^2.$$

We rewrite the equation above as

$$(17) \quad \int |D(v^{q+1} \zeta)|^2 \leq C(n, q) \int v^{2q+2} |D\zeta|^2,$$

replace q by $q - 1$, and apply the Sobolev inequality to get

$$(18) \quad \left(\int v^{2q\kappa} \zeta^{2\kappa} \right)^{\frac{1}{\kappa}} \leq C(n, q) \int v^{2q} |D\zeta|^2 \quad \text{for } q \geq 1$$

with $\kappa = \frac{n}{n-2}$ or $\kappa = 2$ if $n = 2$. We now replace ζ by ζ^β , and we fix ζ so that $|D\zeta|^2$ is bounded pointwise by $C(\tilde{\Omega})$ and $\zeta = 1$ on $\tilde{\Omega}$.

$$(19) \quad \left(\int v^{2q\kappa} \zeta^{2\kappa\beta} \right)^{\frac{1}{2q\kappa}} \leq C \left(\int v^{2q} \zeta^{2\beta-2} \right)^{\frac{1}{2q}}$$

where the constant C now depends on n , q , $\tilde{\Omega}$, and β . Now we iterate the inequality (19) with $q = 1, \kappa, \kappa^2, \dots$ and corresponding $\beta = 2, \beta_1, \beta_2, \dots$, where $\beta_j = 1 + \kappa + \kappa^2 + \dots + 2\kappa^j$. Then we have

$$(20) \quad \left(\int v^{2\kappa^m} \zeta^{2(2\kappa^m + \kappa^{m-1} + \dots + \kappa)} \right)^{\frac{1}{2\kappa^m}} \leq C(n, m, \tilde{\Omega}) \left(\int v^2 \zeta^2 \right)^{\frac{1}{2}}.$$

By (9),

$$\begin{aligned} \left(\int v^{2\kappa^m} \zeta^{2(2\kappa^m + \kappa^{m-1} + \dots + \kappa)} \right)^{\frac{1}{2\kappa^m}} &\leq C(n, m, \tilde{\Omega}) \left(\int u^2 \right)^{\frac{1}{2}} \\ &\leq C(n, m, \tilde{\Omega}, M) \end{aligned}$$

So, on the subdomain $\tilde{\Omega}$ we now have a bound for the Sobolev norm:

$$(21) \quad \|u\|_{W^{1,2\kappa^m}(\tilde{\Omega})} \leq C(n, m, \tilde{\Omega}, M)$$

By the Sobolev Imbedding Theorem, to each α in the statement of the theorem there corresponds an m in equation (21) depending on n and α such that we have a bound on $|u|_{C^{0,\alpha}(\tilde{\Omega})}$. This completes the proof.

Remark: This theorem may be just short of a sharp interior regularity estimate, since the known conical example solutions are at worst Lipschitz. For equation (4) with $0 < \alpha < 1$, a sharp estimate for solutions of the free boundary problem minimizing the variational integral was given by Phillips in [12].

8. LOWER BOUNDS FOR STABLE SOLUTIONS IN LOW DIMENSIONS

Recall that for $n \geq 7$, the radial solutions are unique for their Dirichlet boundary data, therefore maximal and stable. In particular, the conical solution is a stable singular solution for $n \geq 7$. However, for $2 \leq n \leq 6$, the conical solution does not satisfy the stability inequality (6) by Lemma 11 and is not maximal for its boundary data. Thus it cannot be the limit of stable radial solutions and, for $2 \leq n \leq 6$, the stable radial solutions are bounded below by a constant. The next result generalizes this lower bound to all stable solutions in a compact subdomain. Theorem 25 gives the same result on the entire domain.

Theorem 23. *Let $2 \leq n \leq 6$ and let u be a positive stable solution of $\Delta u = \frac{1}{u}$ on the domain Ω with $u \leq M$, and let $\tilde{\Omega} \subset\subset \Omega$ be a compact*

subdomain. Then there is a constant $\delta = \delta(n, \tilde{\Omega}, M) > 0$ such that $u \geq \delta$ on $\tilde{\Omega}$.

Proof. The proof follows from the estimate

$$(22) \quad \int_{\tilde{\Omega}} u^{-p} \leq C(\tilde{\Omega}, p) \quad \text{for } p < 4 + 2\sqrt{2}.$$

Notice that the restriction on p allows for $p \geq n$ as long as $n \leq 6$. For $u > 0$ smooth, we use the stability inequality (6) with the test function ζu^{-q} . Then for $\epsilon > 0$,

$$\begin{aligned} \int u^{-2q-2}\zeta^2 &\leq \int |u^{-q}D\zeta - qu^{-q-1}\zeta Du|^2 \\ &\leq \int u^{-2q}|D\zeta|^2 + 2|q|u^{-2q-1}\zeta|Du||D\zeta| + q^2u^{-2q-2}\zeta^2|Du|^2 \\ (23) \quad &\leq \left(1 + \frac{|q|}{2\epsilon}\right) \int u^{-2q}|D\zeta|^2 + (q^2 + 2|q|\epsilon) \int u^{-2q-2}\zeta^2|Du|^2 \end{aligned}$$

Again, all integrals in this proof are taken over the domain Ω . Notice that in every integral the integrand has compact support in Ω . We will also use the weak form of the equation (8) with the test function $\zeta^2u^{-\beta}$, $\beta > 0$ to get

$$(24) \quad \beta \int u^{-\beta-1}\zeta^2|Du|^2 \leq \int u^{-\beta-1}\zeta^2 + 2 \int u^{-\beta}\zeta|Du||D\zeta|$$

and using Cauchy-Schwartz, for any $\delta > 0$,

$$(25) \quad (\beta - 2\delta) \int u^{-\beta-1}\zeta^2|Du|^2 \leq \int u^{-\beta-1}\zeta^2 + \frac{1}{2\delta} \int u^{-\beta+1}|D\zeta|^2$$

Replacing β by $2q + 1$ and combining with (23), we get for $q > \frac{-1}{2}$ and $\epsilon, \delta > 0$,

$$\begin{aligned} \int u^{-2q-2}\zeta^2 &\leq \left(1 + \frac{|q|}{2\epsilon} + \frac{q^2 + 2|q|\epsilon}{2\delta(2q + 1 - 2\delta)}\right) \int u^{-2q}|D\zeta|^2 + \\ &\quad + \left(\frac{q^2 + 2|q|\epsilon}{2q + 1 - 2\delta}\right) \int u^{-2q-2}\zeta^2 \end{aligned}$$

Then for $1 - \sqrt{2} < q < 1 + \sqrt{2}$ and ϵ and δ small enough depending on q , the coefficient in the last term above is less than one. So,

$$(26) \quad \int \zeta^2 u^{-2q-2} \leq C(q) \int u^{-2q}|D\zeta|^2$$

and now assuming $q > 0$, we replace ζ by ζ^{q+1} :

$$(27) \quad \int \left(\frac{\zeta}{u}\right)^{2q+2} \leq C(q) \int \left(\frac{\zeta}{u}\right)^{2q} |D\zeta|^2$$

Now we use Young's inequality in the form

$$ab \leq \frac{\epsilon^\alpha}{\alpha} a^\alpha + \frac{\alpha-1}{\alpha \epsilon^{\frac{\alpha}{\alpha-1}}} b^{\frac{\alpha}{\alpha-1}}$$

with $\alpha = \frac{q+1}{q}$ and replace q by $\frac{p}{2} - 1$. Then for $2 < p < 4 + 2\sqrt{2}$,

$$(28) \quad \int \left(\frac{\zeta}{u} \right)^p \leq C(p) \int |D\zeta|^p$$

So we get equation (22) for any $p < 4 + 2\sqrt{2}$. We now recall our continuity estimate for stable solutions, that for any $\alpha < 1$, the Hölder norm $|u|_{0,\alpha,\tilde{\Omega}} \leq C(\tilde{\Omega}, \alpha, M)$.

Now let $\tilde{\Omega} \subset \hat{\Omega} \subset \subset \Omega$ with $\text{dist}(\tilde{\Omega}, \partial\hat{\Omega}) > \rho$, so that for any $x \in \tilde{\Omega}$, $B_\rho(x) \subset \hat{\Omega}$. For $x_0 \in \tilde{\Omega}$, let $r = |x - x_0|$ and suppose $u(x_0) = \epsilon$. Then for $x \in B_\rho(x_0) \subset \hat{\Omega}$, $u(x) \leq \epsilon + C(\hat{\Omega}, \alpha, M)r^\alpha$. Then

$$\int_{\hat{\Omega}} u^{-p} \geq \int_{B_\rho(x_0)} u^{-p} \geq \int_{B_\rho(x_0)} (\epsilon + Cr^\alpha)^{-p} \geq n\omega_n \int_0^\rho (\epsilon + Cr^\alpha)^{-p} r^{n-1} dr$$

We may choose α and p large enough that $n - 1 - \alpha p < -1$ for $n \leq 6$. Then for ϵ small enough, we have a contradiction of equation (22) with $\hat{\Omega}$ in place of $\tilde{\Omega}$. This completes the theorem.

We note that we did not need to use the continuity estimate in the above proof. In fact, equation (22) together with the L^p estimates gives an estimate for $u \in C^{1,1-\frac{n}{p}}(\tilde{\Omega})$ for $n \leq 6$ and $p < 4+2\sqrt{2}$. We use a similar method below to get a lower bound on the whole domain. First we present an interesting corollary.

Corollary 24. *There are no complete stable solutions of $\Delta u = \frac{1}{u}$ on all of \mathbb{R}^n for $2 \leq n \leq 6$.*

Proof. Suppose not. From (28), for $2 \leq n \leq 6$ we have

$$\int \left(\frac{\zeta}{u} \right)^n \leq C(n) \int |D\zeta|^n$$

We choose ζ equal to one on the ball B_R , equal to zero outside B_{R^2} , and equal to $2 - \frac{\log|x|}{\log R}$ on $B_{R^2} \setminus B_R$. Then, using the variable $r = |x|$, we have

$$\int_{B_R} \frac{1}{u^n} \leq C \int_R^{R^2} \frac{r^{n-1}}{r^n (\log R)^n} dr \leq \frac{C}{(\log R)^{n-1}}$$

and the result follows letting $R \rightarrow \infty$. We thank Neshan Wickramasekera for pointing out this trick, which also appeared in reference to the Bernstein Theorem in [17].

Theorem 25. *Let $2 \leq n \leq 6$ and let u be a positive stable solution of $\Delta u = \frac{1}{u}$ on the $C^{1,1}$ domain Ω , with boundary data $\varphi \in C^{2,\alpha}(\Omega)$, $|\varphi|_{2,\alpha} \leq M$, and $\varphi \geq \epsilon > 0$. Then there is a constant $\delta = \delta(\Omega, M, \epsilon)$ such that $u \geq \delta$ on Ω .*

The theorem follows from the following lemma.

Lemma 26. *Let $2 \leq n \leq 6$ and let u be a positive stable solution of $\Delta u = \frac{1}{u}$ on the domain Ω , with Lipschitz boundary data φ , and $\varphi \geq \epsilon > 0$. Let $2 \leq p < 4 + 2\sqrt{2}$. Then there is a constant $C(p, |D\varphi|)$ such that*

$$\int_{\Omega} \frac{1}{u^p} \leq \frac{C|\Omega|}{\epsilon^p}.$$

Remark: Notice that in Lemma 26 there is no assumption on the smoothness of the domain.

Proof. Let $\epsilon > 0$ as in the statement and assume $u > 0$ is a stable solution. Let $\eta > 0$ and first consider the stability inequality with the test function $\zeta = (\varphi - u - \eta)_+$:

$$(29) \quad \int \frac{(\varphi - u - \eta)_+^2}{u^2} \leq \int |D(\varphi - u - \eta)_+|^2$$

So,

$$\begin{aligned} \int |D(\varphi - u - \eta)_+|^2 &= \int D\varphi \cdot D(\varphi - u - \eta)_+ - \int Du \cdot D(\varphi - u - \eta)_+ \\ &= \int D\varphi \cdot D(\varphi - u - \eta)_+ + \int \frac{(\varphi - u - \eta)_+}{u} \\ &\leq \int |D\varphi|^2 + \frac{1}{4} \int |D(\varphi - u - \eta)_+|^2 + \frac{1}{2} \int \frac{(\varphi - u - \eta)_+^2}{u^2} + \frac{1}{2} \int 1 \\ &\leq \int |D\varphi|^2 + \frac{3}{4} \int |D(\varphi - u - \eta)_+|^2 + \frac{1}{2} \int 1 \quad (\text{by (29)}) \end{aligned}$$

So,

$$(30) \quad \int |D(\varphi - u - \eta)_+|^2 \leq C \int (1 + |D\varphi|^2)$$

Now we use (27) with the same test function, and replace q by $p/2 - 1$ so that for $2 < p < 4 + 2\sqrt{2}$,

$$(31) \quad \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^p \leq C(p) \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-2} |D(\varphi - u)|^2.$$

Recall from equation (25) that for $\beta > 1$,

$$\int u^{-\beta} \zeta^2 |Du|^2 \leq C \int u^{-\beta} \zeta^2 + C \int u^{2-\beta} |D\zeta|^2.$$

So, assuming $p > 4$, we replace β by $p - 2$ and ζ by $(\varphi - u - \eta)_+^{\frac{p-2}{2}}$ and we have

$$\begin{aligned} \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-2} |Du|^2 &\leq C \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-2} \\ &\quad + C \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-4} |D(\varphi - u - \eta)_+|^2 \end{aligned}$$

We use this with (31) to get

$$\begin{aligned} \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^p + \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-2} |D(\varphi - u)|^2 &\leq \\ C \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-2} + C \int \left(\frac{(\varphi - u - \eta)_+}{u} \right)^{p-4} |D(\varphi - u)|^2 & \end{aligned}$$

where the constant C now depends also on $|D\varphi|$. Now we can apply Young's inequality twice. Then

$$\int \frac{(\varphi - u - \eta)_+^p}{u^p} + \frac{(\varphi - u - \eta)_+^{p-2}}{u^{p-2}} |D(\varphi - u)|^2 \leq C \int 1 + |D(\varphi - u)|^2$$

We then use our estimate (30), and let η tend to zero.

$$\int \left(\frac{(\varphi - u)_+}{u} \right)^p \leq C|\Omega|$$

Now for $\varphi > 2\epsilon$,

$$\frac{1}{u^p} \leq \frac{1}{\epsilon^p} \left(\left(\frac{(\varphi - u)_+}{u} \right)^p + 1 \right)$$

so

$$\int \frac{1}{u^p} \leq \frac{(C+1)|\Omega|}{\epsilon^p}$$

as required.

Remark: Notice that for $p = 2$, we get a stronger result by removing the dependence of C on $|D\varphi|$ and using inequalities (29) and (30). Namely,

$$(32) \quad \int_{\Omega} \frac{1}{u^2} \leq \frac{C}{\epsilon^2} \int_{\Omega} (1 + |D\varphi|^2)$$

where we may assume that φ is merely in $W^{1,2}(\Omega)$.

Proof of Theorem 25: The lemma demonstrates the inequality

$$(33) \quad \int \frac{1}{u^p} \leq C(p, \Omega, \epsilon, |D\varphi|)$$

But of course $\frac{1}{u^p} = (\Delta u)^p$ by the equation, and we can apply the L^p estimates (see [9] 9.14) related to the Calderon–Zygmund Inequality. So,

$$\|u\|_{W^{2,p}} \leq C(p, \Omega, M, \epsilon)$$

and by the extended Sobolev Embedding Theorem,

$$|u|_{C^{1,1-n/p}} \leq C(p, \Omega, M, \epsilon)$$

which in particular implies a uniform Lipschitz bound on u . Then if u achieves the value δ at a point $x_0 \in \Omega$,

$$\int \frac{1}{u^p} \geq \int \frac{1}{(\delta + Cr)^p}$$

where $r = x - x_0$, a contradiction of (33) for $\delta < \delta(p, \Omega, M, \epsilon)$ and $p > n$.

Remark: With this lower bound we in fact have complete regularity of stable solutions for $n \leq 6$. From $u \in C^{1,\alpha}$ and thus (by the lower bound) $\frac{1}{u} \in C^{1,\alpha}$, we can apply Holder estimates to get continuous derivatives of all orders on the interior of the domain.

9. HAUSDORFF DIMENSION OF SINGULAR SETS OF STABLE SOLUTIONS

We use Hausdorff dimension as described in [15].

Theorem 27. Suppose u is a limit of positive stable solutions of $\Delta u = \frac{1}{u}$ on a domain Ω with singular set $A = \{u = 0\}$. Then the Hausdorff dimension $\dim_H(A) \leq n - 4 - 2\sqrt{2}$.

Proof. We will show for any ball B_ρ of radius ρ whose closure is contained in Ω , and any $\beta > n - 4 - 2\sqrt{2}$, that the Hausdorff Measure $\mathcal{H}^\beta(A \cap B_{\rho/2}) < \infty$. First, for any δ with $0 < \delta < \rho/4$, we cover $A \cap B_{\rho/2}$ by cubes Q_j of side length 2δ with disjoint interiors, $j = 1, \dots, N$. Let $p < 4 + 2\sqrt{2}$. By (22), we have

$$\int_{\bigcup Q_j} \frac{1}{u^p} \leq \int_{B_{\rho/2}} \frac{1}{u^p} \leq K < \infty$$

with K independent of δ . By Theorem 22, for any $0 < \alpha < 1$, we have $u(x) \leq C(\text{dist}(x, A))^\alpha$. Thus, assuming all the Q_j intersect A ,

$$\begin{aligned} \int_{Q_j} \frac{C}{u^p} &\geq \int_{Q_j} \frac{1}{(\text{dist}(x, A))^{\alpha p}} \geq \int_{0 < x_i < 2\delta} \frac{1}{(x_1^2 + \dots + x_n^2)^{\alpha p/2}} \geq \\ &\geq \frac{1}{2^n} \int_0^\delta \frac{n\omega_n r^{n-1}}{r^{\alpha p}} dr \geq \frac{n\omega_n}{2^n(n-\alpha p)} \delta^{n-\alpha p} \end{aligned}$$

Choose α so that $\beta = n - \alpha p$. Then we have

$$\mathcal{H}_\delta^\beta(A \cap B_{\rho/2}) \leq C' \sum_{Q_j} \delta^\beta \leq C'' K < \infty$$

independent of δ , and the result follows.

Remark: Recall that the analogous equation (4) with $0 < \alpha < 1$ has actual *minimizers* for the variational problem. In [13], Phillips proved a Hausdorff estimate on the *free boundary* for minimizers. Note that in the free boundary problem, the solution is allowed to vanish completely and not satisfy the PDE on an interior set of positive measure. The technique above gives an estimate on the size of the singular set of a *singular solution* which is a limit of positive solutions satisfying the PDE on the whole domain.

Remark: The major results of this paper extend to the equation (4) with $0 < \alpha < 1$, except for two points. Solutions of (4) which achieve the value zero need not be singular, and the results of sections 8 and 9 are more complicated, with dimensions depending on α .

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